Solutions to Question Sheet 10, Riemann Integration. v1 2019-20

1. Let $f(x) = x^3$ on $[0,1]$ and let $\mathcal{P}_n$ be the arithmetic partition that splits $[0,1]$ into n equal subintervals.

   Evaluate $U(\mathcal{P}_n, f)$ and $L(\mathcal{P}_n, f)$.

   Thus show that f is Riemann integrable on $[0,1]$ and find the value of

$$\int_0^1 x^3 dx.$$

   You may need to recall $\sum_{i=1}^n i^3 = n^2(n+1)^2/4$.

   Solution The arithmetic partition of $[0,1]$ is

$$\mathcal{P}_n = \left\{ \frac{i}{n} : 0 \le i \le n \right\}.$$

   The function $f(x) = x^3$ is increasing on $\mathbb{R}$, so

$$\begin{aligned} M_i &= \sup \left\{ f(x) : \frac{i-1}{n} \le x \le \frac{i}{n} \right\} = \left(\frac{i}{n} \right)^3, \\ m_i &= \inf \left\{ f(x) : \frac{i-1}{n} \le x \le \frac{i}{n} \right\} = \left(\frac{i-1}{n} \right)^3, \end{aligned}$$

   Thus

$$\begin{aligned} U(\mathcal{P}_n, f) &= \sum_{i=1}^n \left(\frac{i}{n} \right)^3 \frac{1}{n} = \frac{1}{n^4} \sum_{i=1}^n i^3, \\ L(\mathcal{P}_n, f) &= \sum_{i=1}^n \left(\frac{i-1}{n} \right)^3 \frac{1}{n} = \frac{1}{n^4} \sum_{i=1}^n (i-1)^3 = \frac{1}{n^4} \sum_{j=1}^{n-1} j^3, \end{aligned}$$

   on writing $j = i-1$. Sum these arithmetic series using the given recollection to get

$$\begin{aligned} U(\mathcal{P}_n, f) &= \frac{1}{n^4} \frac{n^2(n+1)^2}{4} = \frac{1}{4} \left(1 + \frac{1}{n} \right)^2, \\ L(\mathcal{P}_n, f) &= \frac{1}{n^4} \frac{(n-1)^2 n^2}{4} = \frac{1}{4} \left(1 - \frac{1}{n} \right)^2. \end{aligned}$$

From the theory of integration we have,

$$L(\mathcal{P}_n, f) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(\mathcal{P}_n, f)$$

or, in our case,

$$\frac{1}{4}\left(1-\frac{1}{n}\right)^2 \leq \underline{\int_0^1} f(x)\,dx \leq \overline{\int_0^1} f(x)\,dx \leq \frac{1}{4}\left(1+\frac{1}{n}\right)^2$$

for all $n \geq 1$. Let $n \to \infty$ to see that we must have equality in the centre, that is $\underline{\int_0^1} f(x)\,dx = \overline{\int_0^1} f(x)\,dx$. Thus $f(x) = x^3$ is Riemann integrable over $[0,1]$. The common value is $1/4$ so

$$\int_0^1 x^3 dx = \frac{1}{4}.$$

2. i) Integrate $f(x) = x^2$ over $[1,2]$ by using the arithmetic partition of $[1,2]$ into n equal subintervals.

   ii) Integrate $f(x) = x^2$ over $[1,2]$ by using the geometric partition

$$\mathcal{Q}_n = \left\{1, \eta, \eta^2, \eta^3, ..., \eta^n = 2\right\},$$

   where η is the n^{th}-root of 2.

Solution i. The arithmetic partition of $[1,2]$ is

$$\mathcal{P}_n = \left\{1+\frac{i}{n} : 0 \leq i \leq n\right\}.$$

Since $f(x) = x^2$ is increasing on $\mathbb{R}$ we have

$$M_i = \sup\left\{f(x) : 1+\frac{i-1}{n} \leq x \leq 1+\frac{i}{n}\right\} = \left(1+\frac{i}{n}\right)^2,$$

$$m_i = \inf\left\{f(x) : 1+\frac{i-1}{n} \leq x \leq 1+\frac{i}{n}\right\} = \left(1+\frac{i-1}{n}\right)^2,$$

Since the expression for M_i is slightly simpler then that for m_i we consider first the Upper Sum:

$$\begin{aligned}
U(\mathcal{P}_n, f) &= \sum_{i=1}^{n} \left(1+\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right) \\
&= \frac{1}{n}\left(n+\frac{2}{n}\frac{n(n+1)}{2}+\frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{6n^3+6n^2(n+1)+n(n+1)(2n+1)}{6n^3} \\
&= \frac{14n^2+9n+1}{6n^2}.
\end{aligned}$$

For the Lower Sum we wish to reuse work and so attempt to relate the Lower Sum to the Upper Sum.

$$\begin{aligned}
L(\mathcal{P}_n, f) &= \sum_{i=1}^{n}\left(1+\frac{(i-1)}{n}\right)^2\frac{1}{n} = \sum_{j=0}^{n-1}\left(1+\frac{j}{n}\right)^2\frac{1}{n} \\
&= \sum_{j=1}^{n}\left(1+\frac{j}{n}\right)^2\frac{1}{n}+\left(1+\frac{0}{n}\right)^2\frac{1}{n}-\left(1+\frac{n}{n}\right)^2\frac{1}{n} \\
&= U(\mathcal{P}_n, f)+\frac{1}{n}-\frac{4}{n} \\
&= \frac{14n^2+9n+1}{6n^2}-\frac{3}{n} \\
&= \frac{14n^2-9n+1}{6n^2}.
\end{aligned}$$

As in the last question the theory gives

$$\frac{14n^2-9n+1}{6n^2} \le \underline{\int_1^2} f(x)\,dx \le \overline{\int_1^2} f(x)\,dx \le \frac{14n^2+9n+1}{6n^2}.$$

Let $n \to \infty$ to deduce that the Riemann integral exists and

$$\int_1^2 x^2 dx = \frac{7}{3}.$$

ii. Let

$$\mathcal{Q}_n = \left\{\eta^i : 0 \le i \le n\right\}$$

with $\eta = \sqrt[n]{2}$, be the geometric partition of $[1, 2]$. Then

$$\begin{aligned} M_i &= \sup\left\{f(x) : \eta^{i-1} \le x \le \eta^i\right\} = \left(\eta^i\right)^2, \\ m_i &= \inf\left\{f(x) : \eta^{i-1} \le x \le \eta^i\right\} = \left(\eta^{i-1}\right)^2, \end{aligned}$$

Again the expression for M_i is slightly simpler than that for m_i, so consider

$$\begin{aligned} U\left(\mathcal{Q}_n, f\right) &= \sum_{i=1}^{n} \left(\eta^i\right)^2 \left(\eta^i - \eta^{i-1}\right) = \left(1 - \eta^{-1}\right) \sum_{i=1}^{n} \left(\eta^3\right)^i \\ &= \left(1 - \eta^{-1}\right) \frac{\eta^3}{\eta^3 - 1} \left(\eta^{3n} - 1\right) \\ &\qquad \text{on summing the geometric series,} \\ &= \left(1 - \eta^{-1}\right) \frac{7\eta^3}{\eta^3 - 1} \qquad \text{since } \eta^n = 2, \\ &= \frac{7\left(1 - \eta\right)\eta^2}{\left(1 - \eta\right)\left(1 + \eta + \eta^2\right)} = \frac{7\eta^2}{1 + \eta + \eta^2}. \end{aligned}$$

Note in evaluating $U\left(\mathcal{Q}_n, f\right)$ do **not** argue as

$$U\left(\mathcal{Q}_n, f\right) = \sum_{i=1}^{n} \left(\eta^i\right)^2 \left(\eta^i - \eta^{i-1}\right) = \sum_{i=1}^{n} \left(\eta^i\right)^2 \eta^i - \sum_{i=1}^{n} \left(\eta^i\right)^2 \eta^{i-1}.$$

Having two summations simply doubles the chance of making an error.

For the Lower Sum we first express m_i in terms of M_i so we can write the Lower Sum in terms of the Upper Sum and then reuse the calculation above. (No need to do the same work twice.) Thus

$$m_i = \left(\eta^{i-1}\right)^2 = \eta^{-2} \left(\eta^i\right)^2 = \eta^{-2} M_i.$$

So

$$\begin{aligned} L\left(\mathcal{Q}_n, f\right) &= \sum_{i=1}^{n} m_i \left(x_i - x_{i-1}\right) = \eta^{-2} \sum_{i=1}^{n} M_i \left(x_i - x_{i-1}\right) \\ &= \eta^{-2} U\left(\mathcal{Q}_n, f\right) = \frac{7}{1 + \eta + \eta^2}. \end{aligned}$$

Hence

$$\frac{7}{1+\eta+\eta^2} \le \underline{\int_1^2} f(x)\,dx \le \overline{\int_1^2} f(x)\,dx \le \frac{7\eta^2}{1+\eta+\eta^2}.$$

Let $n \to \infty$ when $\eta \to 1$ and we again deduce that the Riemann integral exists and

$$\int_1^2 x^2 dx = \frac{7}{3}.$$

3. Integrate $f(x) = 1/x^3$ over $[2,3]$ by using the geometric partition

$$\mathcal{Q}_n = \left\{2, 2\eta, 2\eta^2, 2\eta^3, ..., 2\eta^n = 3\right\},$$

where η is the n^{th}-root of $3/2$.

Solution Let

$$\mathcal{Q}_n = \left\{2, 2\eta, 2\eta^2, 2\eta^3, ..., 2\eta^n = 3\right\},$$

where η is the n^{th}-root of $3/2$. Then for $1 \le i \le n$ we have

$$[x_{i-1},\, x_i] = \left[2\eta^{i-1},\, 2\eta^i\right].$$

The function $f(x) = x^{-3}$ is decreasing so

$$\begin{aligned} M_i &= \sup\left\{f(x) : 2\eta^{i-1} \le x \le 2\eta^i\right\} = \frac{1}{\left(2\eta^{i-1}\right)^3}, \\ m_i &= \inf\left\{f(x) : 2\eta^{i-1} \le x \le 2\eta^i\right\} = \frac{1}{\left(2\eta^{i}\right)^3}. \end{aligned}$$

Since the expression for m_i is slightly simpler we look first at the Lower Sum.

$$\begin{aligned} L(\mathcal{Q}_n, f) &= \sum_{i=1}^{n} \left(2\eta^i\right)^{-3}\left(2\eta^i - 2\eta^{i-1}\right) = \frac{2}{2^3}\left(1-\eta^{-1}\right)\sum_{i=1}^{n}\left(\eta^{-2}\right)^i \\ &= \frac{1}{4}\left(1-\eta^{-1}\right)\frac{\eta^{-2}}{1-\eta^{-2}}\left(1-\eta^{-2n}\right) \\ &\qquad \text{on summing the geometric series,} \\ &= \frac{1}{4}\left(1-\eta^{-1}\right)\frac{1}{\eta^2-1}\frac{5}{9} \qquad \text{since } \eta^n = 3/2, \\ &= \frac{5}{36}\frac{\eta-1}{\eta}\frac{1}{(\eta+1)(\eta-1)} = \frac{5}{36\eta(1+\eta)}. \end{aligned}$$

For the Upper Sum we have

$$M_i = \frac{1}{\left(2\eta^{i-1}\right)^3} = \frac{\eta^3}{\left(2\eta^i\right)^3} = \eta^3 m_i.$$

Thus

$$\begin{aligned} U\left(\mathcal{Q}_n, f\right) &= \sum_{i=1}^{n} M_i \left(x_i - x_{i-1}\right) = \eta^3 \sum_{i=1}^{n} m_i \left(x_i - x_{i-1}\right) \\ &= \eta^3 L\left(\mathcal{Q}_n, f\right) = \frac{5\eta^2}{36\left(1+\eta\right)}. \end{aligned}$$

Let $n \to \infty$ when $\eta \to 1$ and we again deduce that the Riemann integral exists and

$$\int_2^3 \frac{dx}{x^3} = \frac{5}{72}.$$

4. i) If the function $h : [a, b] \to \mathbb{R}$ is bounded, Riemann integrable and satisfies $h(x) \geq 0$ for all $x \in [a, b]$, show that

$$\int_a^b h(x)\, dx \geq 0.$$

Hint What does $h(x) \geq 0$ for all $x \in [a, b]$ say about any Lower Sum? What does it then say about the Lower Integral of h? Use also the fact that h is Riemann integrable implies that the lower and upper integrals both exist and are equal.

ii) Prove that if the functions f and g, are bounded on $[a, b]$, and satisfy $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\underline{\int_a^b} f \leq \underline{\int_a^b} g \quad \text{and} \quad \overline{\int_a^b} f \leq \overline{\int_a^b} g.$$

iii) Prove that if the Riemann integrable functions f and g satisfy $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g.$$

Solution i. For any partition $\mathcal{P}$ of $[a,b]$, the fact that $h(x) \geq 0$ for all $x \in [a,b]$ means that $L(\mathcal{P}, h) \geq 0$. So

$$\begin{aligned}
\int_a^b h &= \underline{\int_a^b} h \qquad \text{since } h \text{ is integrable,} \\
&= \text{lub}\{L(\mathcal{P},h) : \mathcal{P} \text{ partition}\}, \qquad \text{by definition of } \underline{\int_a^b}, \\
&\geq 0.
\end{aligned}$$

ii. Let a partition $\mathcal{P}$ of $[a,b]$ be given. On any interval $[x_{i-1}, x_i]$, the inequality $f(x) \leq g(x)$ means that

$$\begin{aligned}
M_i^f &= \underset{[x_{i-1},x_i]}{\text{lub}} f(x) \leq \underset{[x_{i-1},x_i]}{\text{lub}} g(x) = M_i^g, \\
m_i^f &= \underset{[x_{i-1},x_i]}{\text{glb}} f(x) \leq \underset{[x_{i-1},x_i]}{\text{glb}} g(x) = m_i^g.
\end{aligned}$$

Thus

$$L(\mathcal{P}, f) \leq L(\mathcal{P}, g) \quad \text{and} \quad U(\mathcal{P}, f) \leq U(\mathcal{P}, g) \tag{1}$$

for all $\mathcal{P}$.

By definition $\underline{\int_a^b} g$ is *an* upper bound for all $L(\mathcal{P}, g)$ as $\mathcal{P}$ varies. From (1) we then get that $\underline{\int_a^b} g$ is an upper bound for $\{L(\mathcal{P}, f) : \mathcal{P}\}$. Yet by definition $\underline{\int_a^b} f$ is the *least* of all upper bounds of this set, and so

$$\underline{\int_a^b} f \leq \underline{\int_a^b} g. \tag{2}$$

Similarly, $\overline{\int_a^b} f$ is *a* lower bound for all $U(\mathcal{P}, f)$ as $\mathcal{P}$ varies. Again from (1) we then get that $\overline{\int_a^b} f$ is a lower bound for $\{U(\mathcal{P}, g) : \mathcal{P}\}$. Yet by definition $\overline{\int_a^b} g$ is the *greatest* of all lower bounds of this set, and so

$$\overline{\int_a^b} g \geq \overline{\int_a^b} f.$$

iii. The fact that f and g are Riemann integrable gives

$$\begin{aligned}
\int_a^b f &= \underline{\int_a^b} f && \text{since } f \text{ is Riemann integrable,} \\
&\le \underline{\int_a^b} g && \text{by } (2)\,, \\
&= \int_a^b g && \text{since } g \text{ is Riemann integrable.}
\end{aligned}$$

5. Integrate $f(x) = x^2 - x$ over $[2,5]$ by using

   i) the arithmetic partition of $[2,5]$ into n equal length subintervals and

   ii) the geometric partition of $[2,5]$ into n intervals.

Solution i. Let $f(x) = x^2 - x$ and

$$\mathcal{P}_n = \left\{ 2 + \frac{3i}{n} : 0 \le i \le n \right\},$$

an arithmetic partition of $[2,5]$. The function f is increasing for $x > 1/2$ and thus on this interval. Hence

$$\begin{aligned}
M_i &= \sup\left\{ f(x) : 2 + \frac{3\,(i-1)}{n} \le x \le 2 + \frac{3i}{n} \right\} \\
&= \left(2 + \frac{3i}{n} \right)^2 - \left(2 + \frac{3i}{n} \right), \\
m_i &= \inf\left\{ f(x) : 2 + \frac{3\,(i-1)}{n} \le x \le 2 + \frac{3i}{n} \right\} \\
&= \left(2 + \frac{3\,(i-1)}{n} \right)^2 - \left(2 + \frac{3\,(i-1)}{n} \right),
\end{aligned}$$

Consider first the Upper Sum:

$$\begin{aligned}
U(\mathcal{P}_n, f) &= \sum_{i=1}^{n} \left\{ \left(2+\frac{3i}{n}\right)^2 - \left(2+\frac{3i}{n}\right) \right\} \frac{3}{n} \\
&= \frac{3}{n}\sum_{i=1}^{n}\left(2+\frac{9i}{n}+\frac{9i^2}{n^2}\right) \\
&= \frac{3}{n}\left(2n+\frac{9}{n}\frac{n(n+1)}{2}+\frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{(12n^3+27n^2(n+1)+9n(n+1)(2n+1))}{2n^3} \\
&= \frac{57n^2+54n+9}{2n^2}.
\end{aligned}$$

For the Lower Sum

$$L(\mathcal{P}_n, f) = \sum_{i=1}^{n} \left\{ \left(2+\frac{3(i-1)}{n}\right)^2 - \left(2+\frac{3(i-1)}{n}\right) \right\} \frac{3}{n}.$$

Change variable from i to $j = i-1$ so the sum now runs from 0 to $n-1$:

$$L(\mathcal{P}_n, f) = \sum_{j=0}^{n-1} \left\{ \left(2+\frac{3j}{n}\right)^2 - \left(2+\frac{3j}{n}\right) \right\} \frac{3}{n}.$$

Next, express this in terms of $U(\mathcal{P}_n, f)$,

$$\begin{aligned}
L(\mathcal{P}_n, f) &= U(\mathcal{P}_n, f) + \left\{ \left(2+\frac{3\times 0}{n}\right)^2 - \left(2+\frac{3\times 0}{n}\right) \right\} \frac{3}{n} \\
&\qquad - \left\{ \left(2+\frac{3\times n}{n}\right)^2 - \left(2+\frac{3\times n}{n}\right) \right\} \frac{3}{n} \\
&= U(\mathcal{P}_n, f) + \frac{6}{n} - \frac{60}{n} \\
&= \frac{57n^2+54n+9}{2n^2} - \frac{54}{n} \\
&= \frac{57n^2-54n+9}{2n^2}.
\end{aligned}$$

From the theory we have

$$\begin{aligned} \frac{57n^2-54n+9}{2n^2} &\leq \underline{\int_2^5} f(x)\,dx \\ &\leq \overline{\int_2^5} f(x)\,dx \leq \frac{57n^2+54n+9}{2n^2}. \end{aligned}$$

Let $n \to \infty$ to deduce that the Riemann integral exists and

$$\int_2^5 \left(x^2 - x\right) dx = \frac{57}{2}.$$

ii) Let

$$\mathcal{Q}_n = \left\{2\eta^i : 0 \leq i \leq n\right\}$$

with $\eta = \sqrt[n]{5/2}$, be the geometric partition of $[2,5]$. Then, since f is increasing on $[2,5]$,

$$\begin{aligned} M_i &= \sup\left\{f(x) : 2\eta^{i-1} \leq x \leq 2\eta^i\right\} = \left(2\eta^i\right)^2 - 2\eta^i, \\ m_i &= \inf\left\{f(x) : 2\eta^{i-1} \leq x \leq 2\eta^i\right\} = \left(2\eta^{i-1}\right)^2 - 2\eta^{i-1}. \end{aligned}$$

Thus

$$
\begin{aligned}
U\left(\mathcal{Q}_n, f\right) &= \sum_{i=1}^{n} \left(\left(2\eta^i\right)^2 - 2\eta^i\right)\left(2\eta^i - 2\eta^{i-1}\right) \\
&= \left(1-\eta^{-1}\right)\left(8\sum_{i=1}^{n}\left(\eta^3\right)^i - 4\sum_{i=1}^{n}\left(\eta^2\right)^i\right) \\
&= \left(1-\eta^{-1}\right)\left(8\frac{\eta^3}{\eta^3-1}\left(\eta^{3n}-1\right) - 4\frac{\eta^2}{\eta^2-1}\left(\eta^{2n}-1\right)\right) \\
&\qquad \text{on summing the geometric series,} \\
&= 117\left(1-\eta^{-1}\right)\frac{\eta^3}{\eta^3-1} - 21\left(1-\eta^{-1}\right)\frac{\eta^2}{\eta^2-1} \\
&\qquad \text{since } \eta^n = 5/2, \\
&= \frac{117\left(1-\eta\right)\eta^2}{\left(1-\eta\right)\left(1+\eta+\eta^2\right)} - 21\left(1-\eta\right)\frac{\eta}{\left(1-\eta\right)\left(1+\eta\right)} \\
&= 117\frac{\eta^2}{1+\eta+\eta^2} - 21\frac{\eta}{1+\eta}.
\end{aligned}
$$

For the Lower Sum we have

$$
\begin{aligned}
L\left(\mathcal{Q}_n, f\right) &= \sum_{i=1}^{n}\left(\left(2\eta^{i-1}\right)^2 - 2\eta^{i-1}\right)\left(2\eta^i - 2\eta^{i-1}\right) \\
&= \left(1-\eta^{-1}\right)\left(\frac{8}{\eta^2}\sum_{i=1}^{n}\left(\eta^3\right)^i - \frac{4}{\eta}\sum_{i=1}^{n}\left(\eta^2\right)^i\right) \\
&= 117\frac{1}{1+\eta+\eta^2} - 21\frac{1}{1+\eta}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{117}{1+\eta+\eta^2} - 21\frac{1}{1+\eta} &\le \underline{\int_1^2} f(x)\,dx \\
&\le \overline{\int_1^2} f(x)\,dx \le 117\frac{\eta^2}{1+\eta+\eta^2} - 21\frac{\eta}{1+\eta}.
\end{aligned}
$$

Let $n \to \infty$ when $\eta \to 1$ and we again deduce that the Riemann integral exists and

$$\int_1^2 \left(x^2 - x\right) dx = \frac{117}{3} - \frac{21}{2} = \frac{57}{2}.$$

6. **Definition** *If f is continuous on $(a,b)$and F is continuous on $[a,b]$ and and differentiable on (a,b) with $F'(x) = f(x)$ for all $x \in (a,b)$ then F is a* ***primitive*** *for f.*

   Find primitives for

   $$\text{(i)}\ \frac{1}{\sqrt{1-x^2}}, \qquad \text{(ii)}\ \frac{x}{\sqrt{1-x^2}}, \qquad \text{(iii)}\ \frac{1}{\sqrt{1+x^2}}.$$

   $$\text{iv)}\ \frac{x}{\sqrt{1+x^2}}, \qquad \text{v)}\ \frac{1}{1+x^2}, \qquad \text{vi)}\ \frac{x}{1+x^2}.$$

   Solution A primitive of

   i. $1/\sqrt{1-x^2}$ is $\arcsin x$, by Question 8ii, Sheet 7,

   ii. $x/\sqrt{1-x^2}$ is $-\sqrt{1-x^2}$

   iii. $1/\sqrt{1+x^2}$ is $\sinh^{-1} x$, by Question 10i, Sheet 7,

   iv. $x/\sqrt{1+x^2}$ is $\sqrt{1+x^2}$.

   v. $1/\left(1+x^2\right)$ is $\arctan x$, by Question 8iii, Sheet 7,

   vi. $x/\left(1+x^2\right)$ is $\ln\sqrt{(1+x^2)}$.

7. The **Fundamental Theorem of Calculus** says, in part, that *if f is continuous on (a,b) then $F(x) = \int_a^x f(t)\,dt$ is a primitive for $f(x)$ on (a,b).*

   Prove that $\ln x$, defined earlier as the inverse of e^x, satisfies

   $$\ln x = \int_1^x \frac{dt}{t}$$

   for all $x > 0$.

   Hint: Find two primitives for $f : (0,\infty) \to \mathbb{R}, x \longmapsto 1/x$ and note that primitives are unique up to a constant.

Solution From the notes we know that (as an example of differentiating inverse functions)

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

for $x > 0$ and so $\ln x$ is **a** primitive for $1/x$ in this range. But, since $1/t$ is Riemann integrable and continuous on $(0, \infty)$ we know, from the Fundamental Theorem of Calculus, that

$$F(x) := \int_1^x \frac{dt}{t} \; \left(= - \int_x^1 \frac{dt}{t} \text{ if } x < 1 \right)$$

is also **a** primitive for $1/x$. Primitives are unique up to a constant, so

$$\ln x = \int_1^x \frac{dt}{t} + C$$

for some constant C. Put $x = 1$ to find that $C = 0$.

Solutions to Additional Questions

8. Integrate $f(x) = x^2 - 6x + 10$ over $[2,5]$ using the arithmetic partition of $[2,5]$ into $3n$ equal length subintervals.

   Note how we look at $\mathcal{P}_{3n}$ and not $\mathcal{P}_n$, ask yourself why.

   Solution Let $f(x) = x^2 - 6x + 10$ on $[2,5]$. This time $f'(x) = 2x - 6$ so f increases for $x > 3$ and decreases for $x < 3$.

   Look at the partition

$$\mathcal{P}_{3n} = \left\{2 + \frac{3i}{3n} : 0 \le i \le 3n\right\} = \left\{2 + \frac{i}{n} : 0 \le i \le 3n\right\}.$$

   We have chosen $3n$ instead of n so that one of the points in the partition is $x = 3$, (when $i = n$) where the function has a turning point. Note that the width of the intervals in the partition is $1/n$.

   Because of the minimum at $x = 3$, i.e. $i = n$, we have

$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\} = \begin{cases} f(x_{i-1}) & \text{for } 1 \le i \le n \\ f(x_i) & \text{for } n+1 \le i \le 3n. \end{cases}$$

   Similarly

$$m_i = \sup\{f(x) : x_{i-1} \le x \le x_i\} = \begin{cases} f(x_i) & \text{for } 1 \le i \le n \\ f(x_{i-1}) & \text{for } n+1 \le i \le 3n. \end{cases}$$

   Note that

$$f(x_i) = f\left(2 + \frac{i}{n}\right) = \frac{i^2}{n^2} - 2\frac{i}{n} + 2,$$

   and so

$$f(x_{i-1}) = \frac{(i-1)^2}{n^2} - 2\frac{(i-1)}{n} + 2.$$

Hence

$$\begin{aligned}
U\left(\mathcal{P}_{3n}, f\right) &= \sum_{i=1}^{n}\left(\frac{(i-1)^2}{n^2} - 2\frac{(i-1)}{n} + 2\right)\frac{1}{n} \\
&\quad + \sum_{i=n+1}^{3n}\left(\frac{i^2}{n^2} - 2\frac{i}{n} + 2\right)\frac{1}{n} \\
&= \frac{1}{n^3}\sum_{i=1}^{n-1} i^2 - \frac{2}{n^2}\sum_{i=1}^{n-1} i + 2 \\
&\quad + \frac{1}{n^3}\sum_{i=n+1}^{3n} i^2 - \frac{2}{n^2}\sum_{i=n+1}^{3n} i + 4.
\end{aligned}$$

We can combine two pairs of summations, noting that the $i = n$ term is missing in both. So

$$\begin{aligned}
U\left(\mathcal{P}_{3n}, f\right) &= \frac{1}{n^3}\left(\sum_{i=1}^{3n} i^2 - n^2\right) - \frac{2}{n^2}\left(\sum_{i=1}^{3n} i - n\right) + 6 \\
&= \frac{1}{n^3}\left(9n^3 + \frac{7}{2}n^2 + \frac{1}{2}n\right) - \frac{2}{n^2}\left(\frac{9}{2}n^2 + \frac{1}{2}n\right) + 6 \\
&= \frac{12n^2 + 5n + 1}{2n^2}
\end{aligned}$$

Similarly

$$\begin{aligned}
L\left(\mathcal{P}_{3n}, f\right) &= \sum_{i=1}^{n}\left(\frac{i^2}{n^2} - 2\frac{i}{n} + 2\right)\frac{1}{n} \\
&\quad + \sum_{i=n+1}^{3n}\left(\frac{(i-1)^2}{n^2} - 2\frac{(i-1)}{n} + 2\right)\frac{1}{n} \\
&= \frac{12n^2 - 5n + 1}{2n^2}
\end{aligned}$$

It matters not that we have $3n$ in place of n in

$$L\left(\mathcal{P}_{3n}, f\right) \le \underline{\int_2^5} f \le \overline{\int_2^5} f \le U\left(\mathcal{P}_{3n}, f\right).$$

Thus

$$\frac{12n^2 - 5n + 1}{2n^2} \le \underline{\int_2^5} f(x)\,dx \le \overline{\int_2^5} f(x)\,dx \le \frac{12n^2 + 5n + 1}{2n^2}.$$

Let $n \to \infty$ to deduce that the Riemann integral exists and

$$\int_2^5 \left(x^2 - 6x + 10\right) dx = 6.$$

Note In this proof we have essentially calculated $\int_2^3 f$, $\int_3^5 f$ and added the results together. That you can do this is a result we have not had time to cover in the course.

9. Let $f : [0,1] \to \mathbb{R}$ be given by $f(0) = 0$ and, for $x \in (0,1]$,

$$f(x) = \frac{1}{n} \text{ where } n \text{ is the largest integer satisfying } n \le \frac{1}{x}.$$

Draw the graph of f. Show that f is monotonic on $[0,1]$.

Deduce that f is Riemann integrable on $[0,1]$.

Find

$$\int_0^1 f.$$

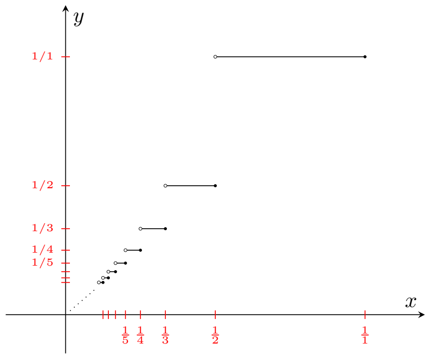
Hint. First calculate the integral over $[1/N, 1]$ for any $N \ge 1$. Then use this in evaluating the upper and lower integrals of f over $[0,1]$.

Solution Let $0 \le x < y \le 1$ be given. Write n_x for the largest integer $n_x \le 1/x$ so $f(x) = 1/n_x$. Similarly n_y is the largest integer $\le 1/y$. Then

$$x < y \implies \frac{1}{y} < \frac{1}{x} \implies n_y \le n_x \implies f(x) = \frac{1}{n_x} \le \frac{1}{n_y} = f(y).$$

Hence f is a monotonic (in fact, increasing) function.

Graph of $y = f(x)$:



It can be shown that any monotonic function is Riemann integrable. Here, though, we will not assume this but first note that f is Riemann integrable over the interval $[1/N, 1]$ for any $N \geq 1$. In fact

$$\begin{aligned}\int_{1/N}^{1} f(x)\,dx &= \sum_{j=1}^{N-1} \int_{\frac{1}{j+1}}^{\frac{1}{j}} \frac{1}{j} = \sum_{j=1}^{N-1} \frac{1}{j}\left(\frac{1}{j} - \frac{1}{j+1}\right) \\ &= \sum_{j=1}^{N-1} \frac{1}{j^2} - \sum_{j=1}^{N-1} \frac{1}{j\,(j+1)}.\end{aligned}$$

Here we have a 'telescoping' series,

$$\begin{aligned}\sum_{j=1}^{N-1} \frac{1}{j\,(j+1)} &= \sum_{j=1}^{N-1} \left(\frac{1}{j} - \frac{1}{j+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{N-1} - \frac{1}{N}\right) \\ &= 1 - \frac{1}{N}.\end{aligned}$$

So

$$\int_{1/N}^{1} f(x)\,dx = \sum_{j=1}^{N-1} \frac{1}{j^2} - 1 + \frac{1}{N}.$$

We cannot justify letting $N \to \infty$, instead we examine the upper and lower integrals of f.

First $f \geq 0$ implies

$$\underline{\int_0^1} f(x)\,dx \geq \underline{\int_{1/N}^1} f(x)\,dx = \int_{1/N}^1 f(x)\,dx,$$

the last step following from f being Riemann integrable over the interval of integration.

For an upper bound we note that if $0 < x < 1/N$ then $N < 1/x$. So if N_x is the largest integer $\leq 1/x$ we have $N_x \geq N$. Yet by definition $f(x) = 1/N_x$ and so $f(x) \leq 1/N$. That is,

$$0 < x < \frac{1}{N} \implies f(x) \leq \frac{1}{N}.$$

Hence

$$\begin{aligned}
\overline{\int_0^1} f(x)\,dx &= \overline{\int_0^{1/N}} f(x)\,dx + \overline{\int_{1/N}^1} f(x)\,dx \\
&\leq \overline{\int_0^{1/N}} \frac{1}{N} + \overline{\int_{1/N}^1} f(x)\,dx = \frac{1}{N^2} + \int_{1/N}^1 f(x)\,dx
\end{aligned}$$

Combining we have

$$\int_{1/N}^1 f(x)\,dx \leq \underline{\int_0^1} f(x)\,dx \leq \overline{\int_0^1} f(x)\,dx \leq \frac{1}{N^2} + \int_{1/N}^1 f(x)\,dx.$$

That is,

$$\begin{aligned}
\sum_{j=1}^{N-1} \frac{1}{j^2} - 1 + \frac{1}{N} &\leq \underline{\int_0^1} f(x)\,dx \\
&\leq \overline{\int_0^1} f(x)\,dx \leq \sum_{j=1}^{N-1} \frac{1}{j^2} - 1 + \frac{1}{N} + \frac{1}{N^2}.
\end{aligned}$$

Now let $N \to \infty$, concluding that the lower and upper integrals agree and so f is Riemann integrable over $[0, 1]$. Further, the value of the integral is the common value,

$$\int_0^1 f(x)\,dx = \sum_{j=1}^{\infty} \frac{1}{j^2} - 1 = \frac{\pi^2}{6} - 1.$$